The Generalized Alignment Index (GALI) Method of Chaos Detection: Theory and Applications

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Outline

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- **Generalized ALignment Index – GALI**
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Autonomous Hamiltonian systems

Consider an N degree of freedom autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\underbrace{q_1,q_2,\ldots,q_N}_{\text{positions}}, \underbrace{p_1,p_2,\ldots,p_N}_{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0)=(q_1(0), q_2(0),\ldots,q_N(0), p_1(0), p_2(0),\ldots,p_N(0))$$

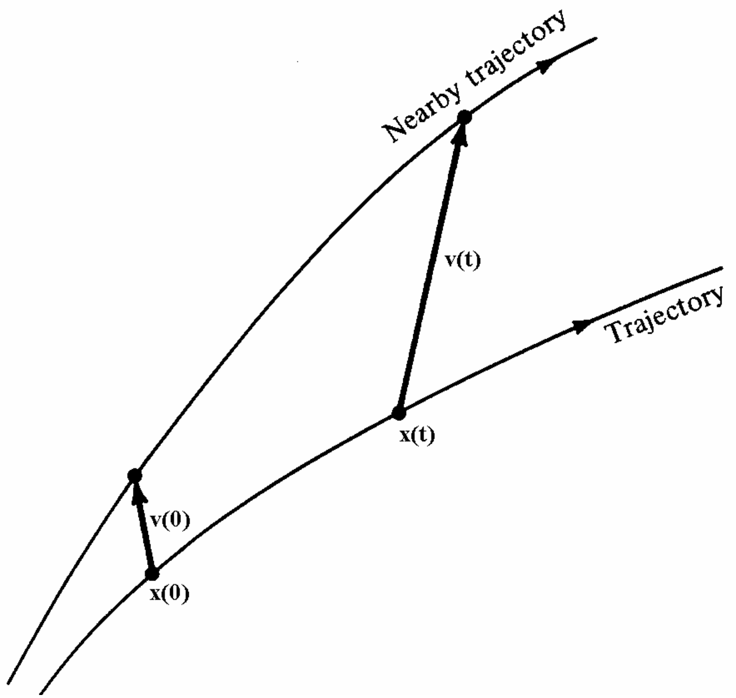
is governed by the Hamilton's equations of motion

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Variational Equations

We use the notation $\mathbf{x} = (q_1,q_2,\ldots,q_N,p_1,p_2,\ldots,p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (dx_1, dx_2,\ldots, dx_n)^T \text{ , with } n=2N$$



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

The time evolution of v is given by the so-called **variational equations:**

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J}\cdot\mathbf{P}\cdot\mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}, \quad P_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_j} \quad i,j = 1,2,\ldots,n$$

Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\sigma(x(0), v(0)) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|v(t)\|}{\|v(0)\|}$$

Computation of the Maximal Lyapunov Exponent

Due to the exponential growth of v(t) (and of d(t)=||v(t)||) we renormalize v(t) from time to time.

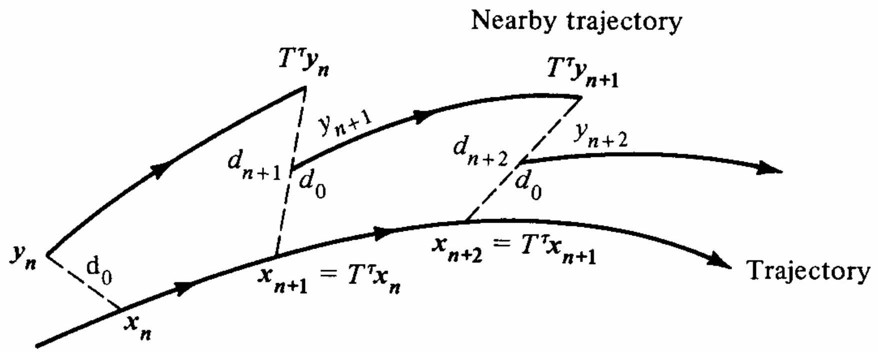


Figure 5.6. Numerical calculation of the maximal Liapunov characteristic exponent. Here $y = x + v$ and τ is a finite interval of time (after Benettin *et al.*, 1976).

Then the Maximal Lyapunov exponent is computed as

$$\sigma_1 = \lim_{n\to\infty} \frac{1}{n\tau} \sum_{i=1}^{n} \ln d_i$$

Maximal Lyapunov Exponent

$\sigma_1=0 \rightarrow$ **Regular motion**
$\sigma_1\neq0 \rightarrow$ **Chaotic motion**

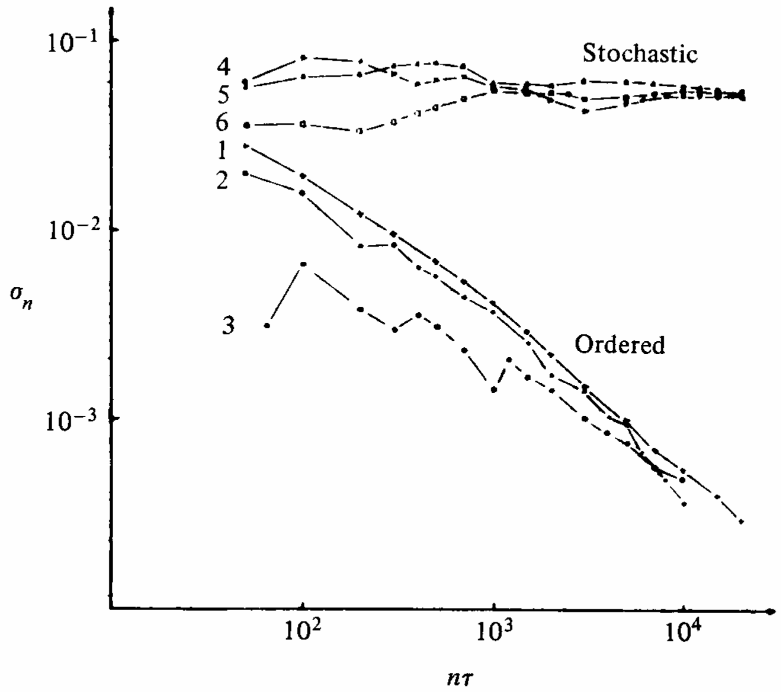


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

If we start with more than one linearly independent deviation vectors they will align to the direction defined by the largest Lyapunov exponent.

Definition of Smaller Alignment Index (SALI)

Consider the n-dimensional phase space of a conservative dynamical system (symplectic map or Hamiltonian flow).

An orbit in that space with initial condition :

$$P(0)=(x_1(0), x_2(0),\ldots,x_n(0))$$

and a deviation vector

$$v(0)=(dx_1(0), dx_2(0),\ldots, dx_n(0))$$

The evolution in time (in maps the time is discrete and is equal to the number N of the iterations) of a deviation vector is defined by:

- **the variational equations (for Hamiltonian flows) and**
- **the equations of the tangent map (for mappings)**

Definition of SALI

We follow the evolution in time of two different initial deviation vectors ($v_1(0)$, $v_2(0)$), and define SALI (Skokos, 2001, J. Phys. A, 34, 10029) as:

$$\mathrm{SALI}(t) = \min\left\{ \left\| \hat{v}_1(t) + \hat{v}_2(t) \right\|, \left\| \hat{v}_1(t) - \hat{v}_2(t) \right\| \right\}$$

where

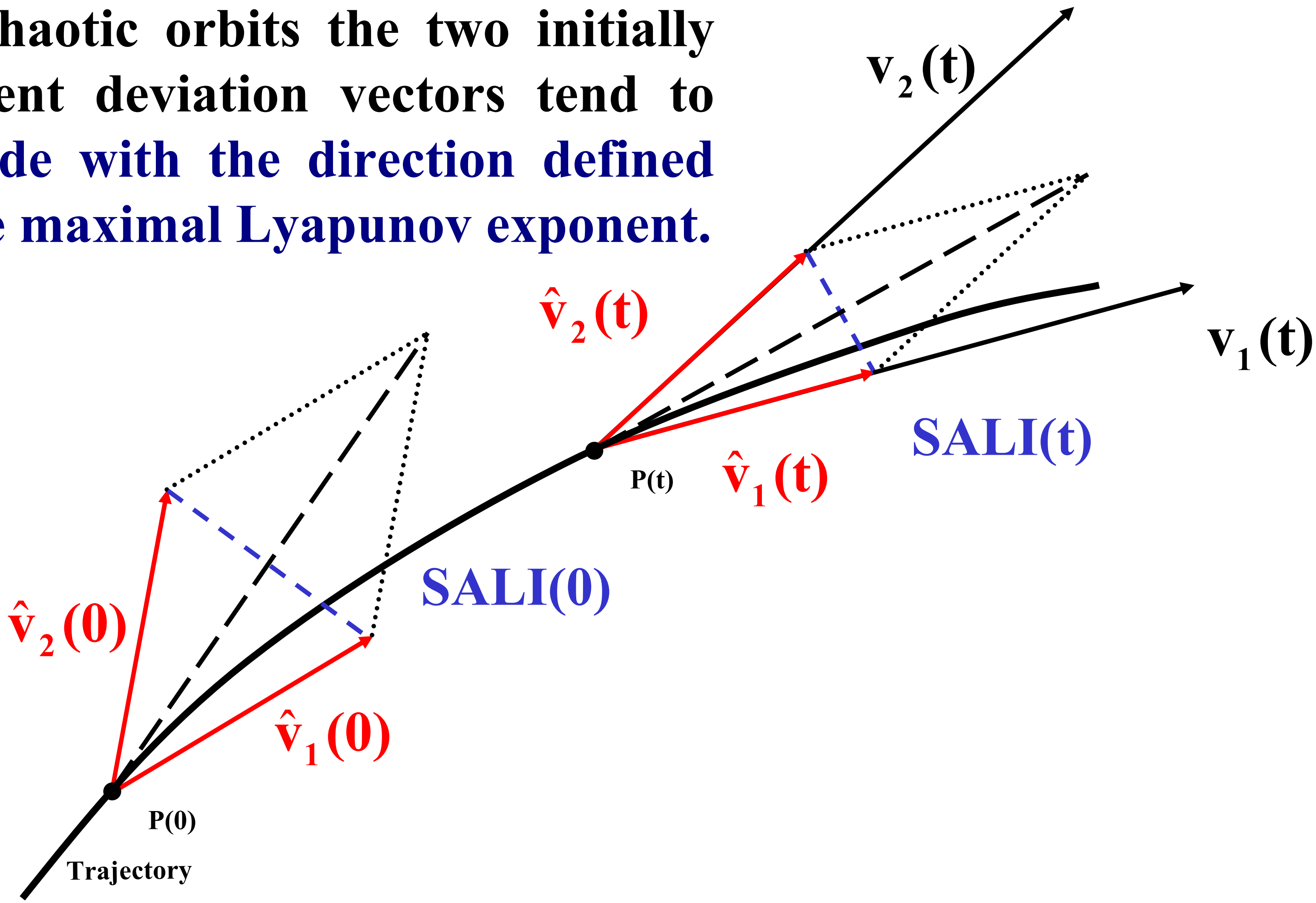
$$\hat{v}_1(t) = \frac{v_1(t)}{\left\| v_1(t) \right\|}$$

When the two vectors become collinear

$$\mathrm{SALI}(t) \rightarrow 0$$

Behavior of SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximal Lyapunov exponent.

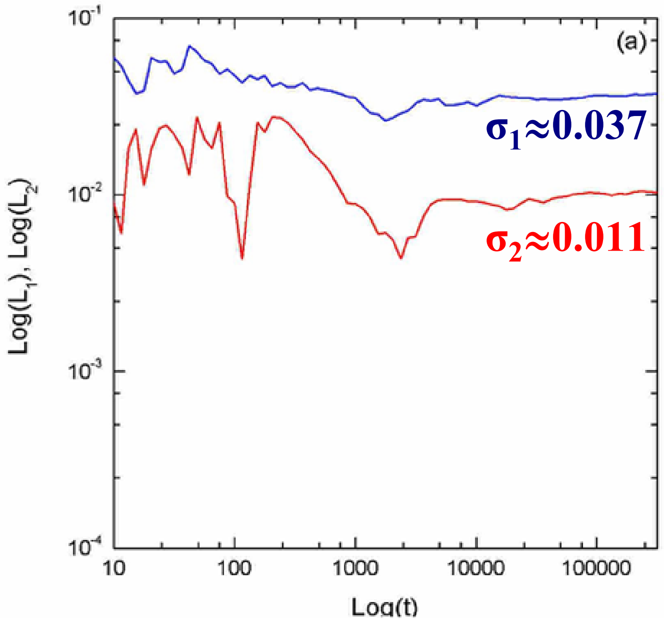


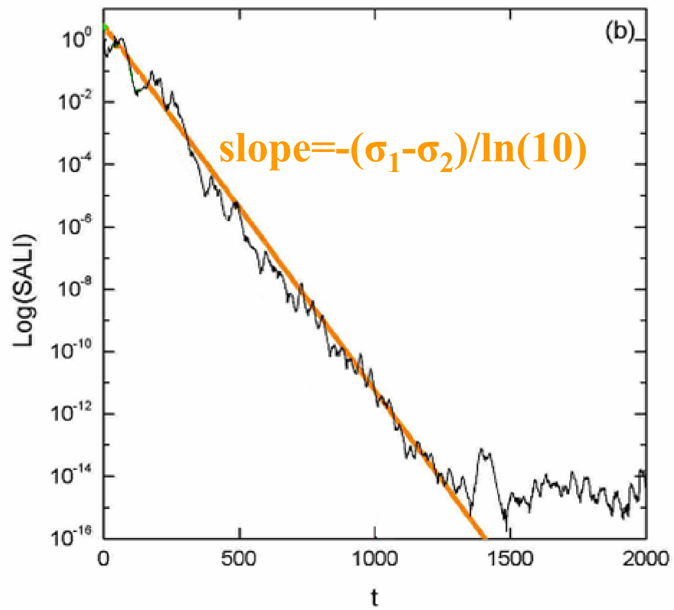
Behavior of SALI for chaotic motion

We test the validity of the approximation $SALI \propto e^{-(\sigma_1-\sigma_2)t}$ (Skokos et al., 2004, J. Phys. A, 37, 6269) for a chaotic orbit of the 3D Hamiltonian

$$H = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

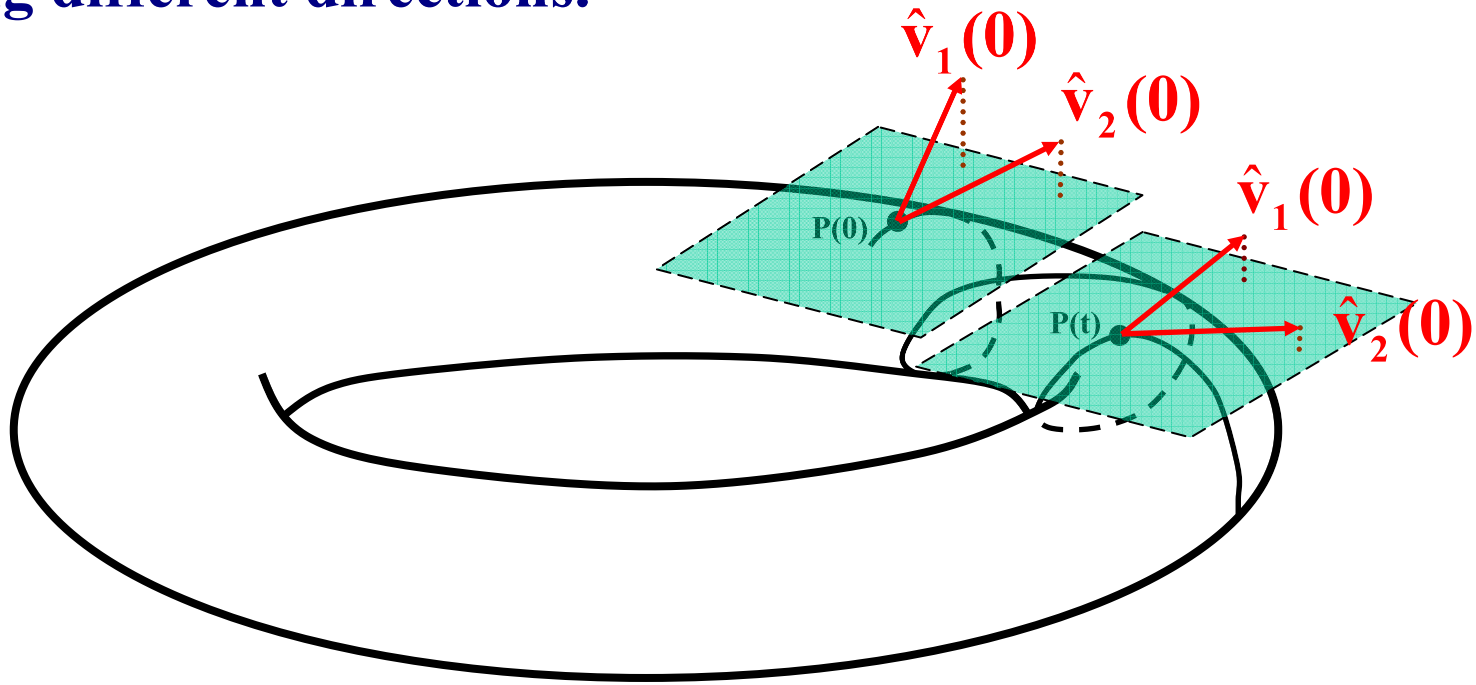
with ω_1=1, ω_2=1.4142, ω_3=1.7321, H=0.09





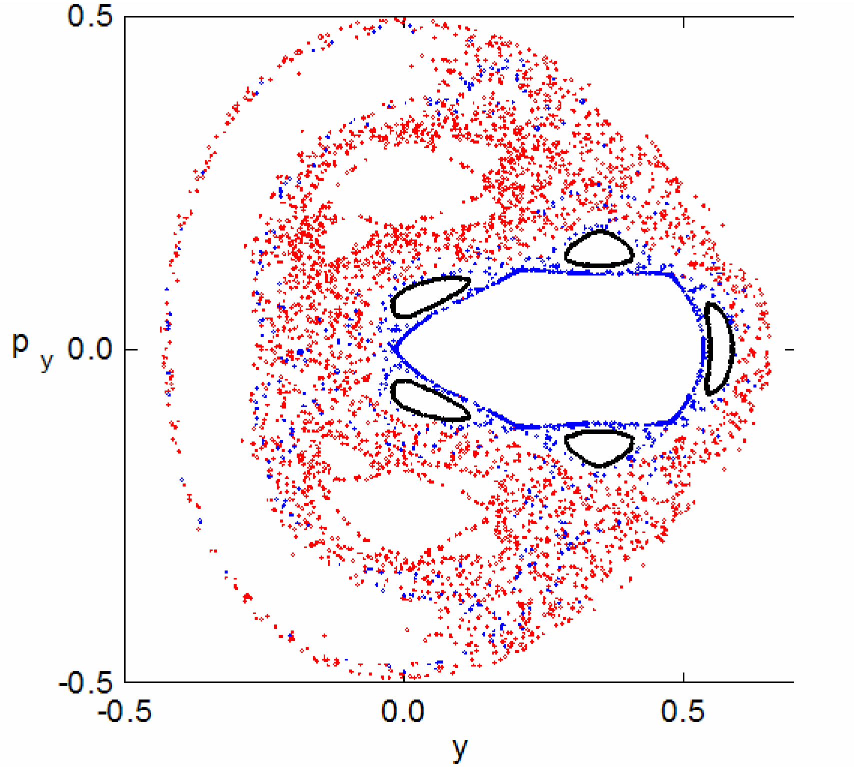
Behavior of SALI for regular motion

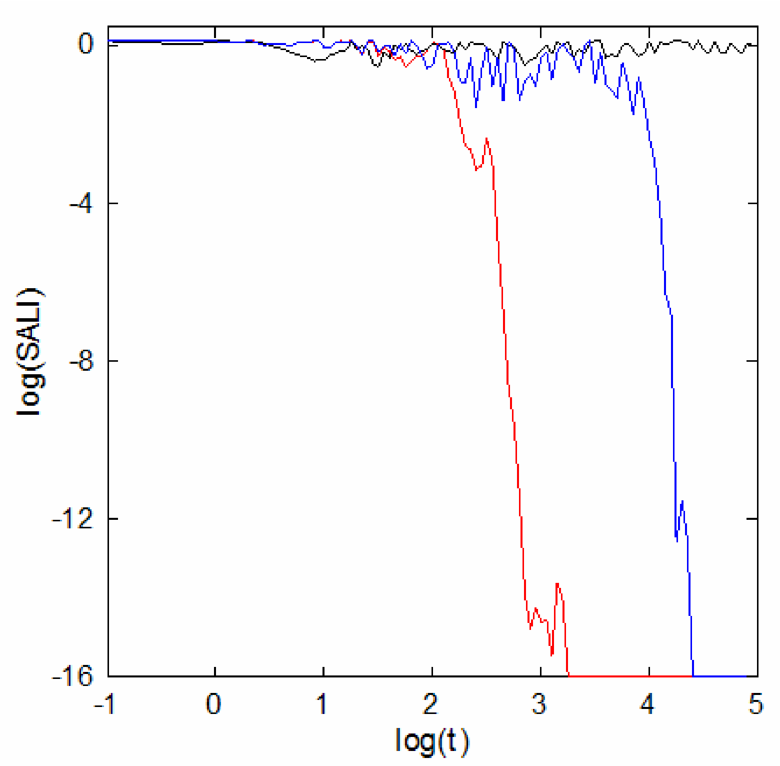
Regular motion occurs on a torus and two different initial deviation vectors become tangent to the torus, generally having different directions.



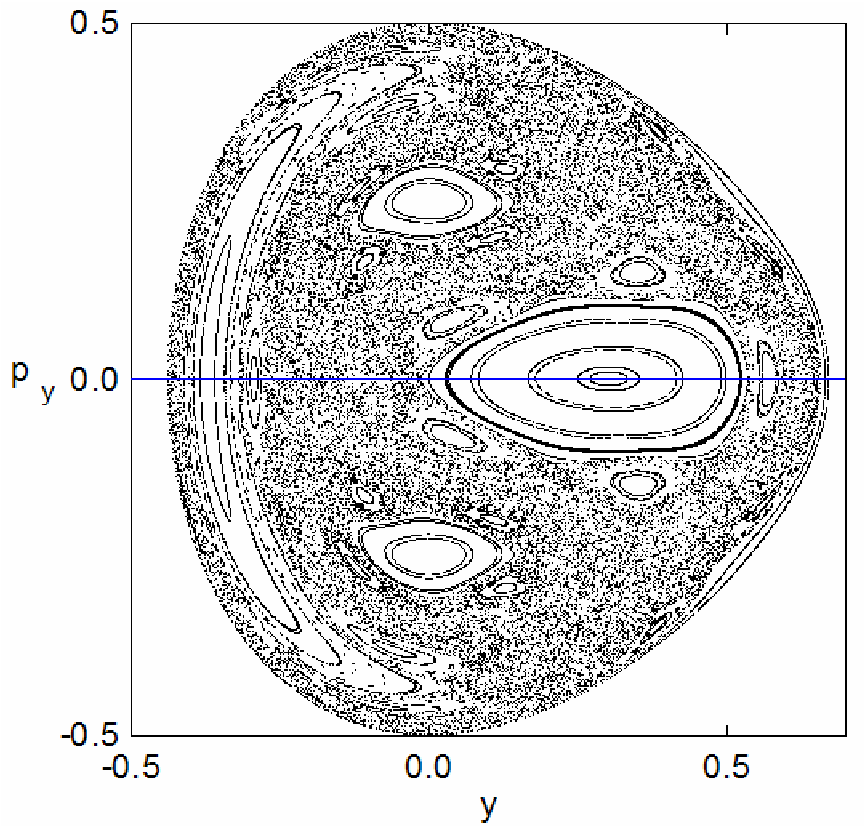
Applications – Hénon-Heiles system

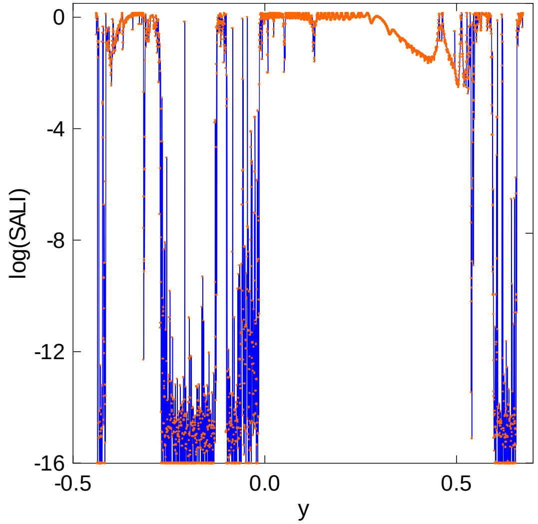
For E=1/8 we consider the orbits with initial conditions:
Ordered orbit, x=0, y=0.55, p_x=0.2417, p_y=0
Chaotic orbit, x=0, y=-0.016, p_x=0.49974, p_y=0
Chaotic orbit, x=0, y=-0.01344, p_x=0.49982, p_y=0



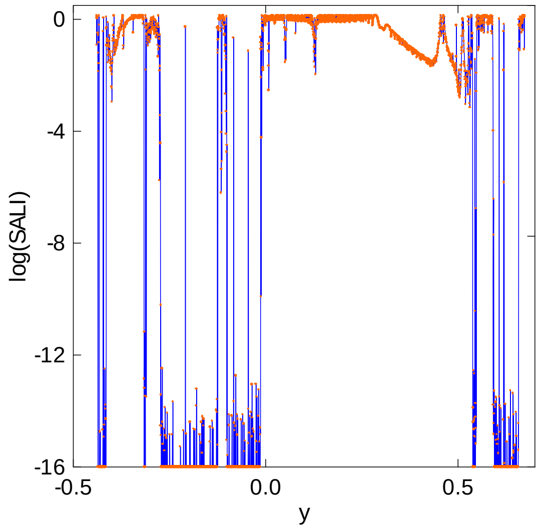


Applications – Hénon-Heiles system



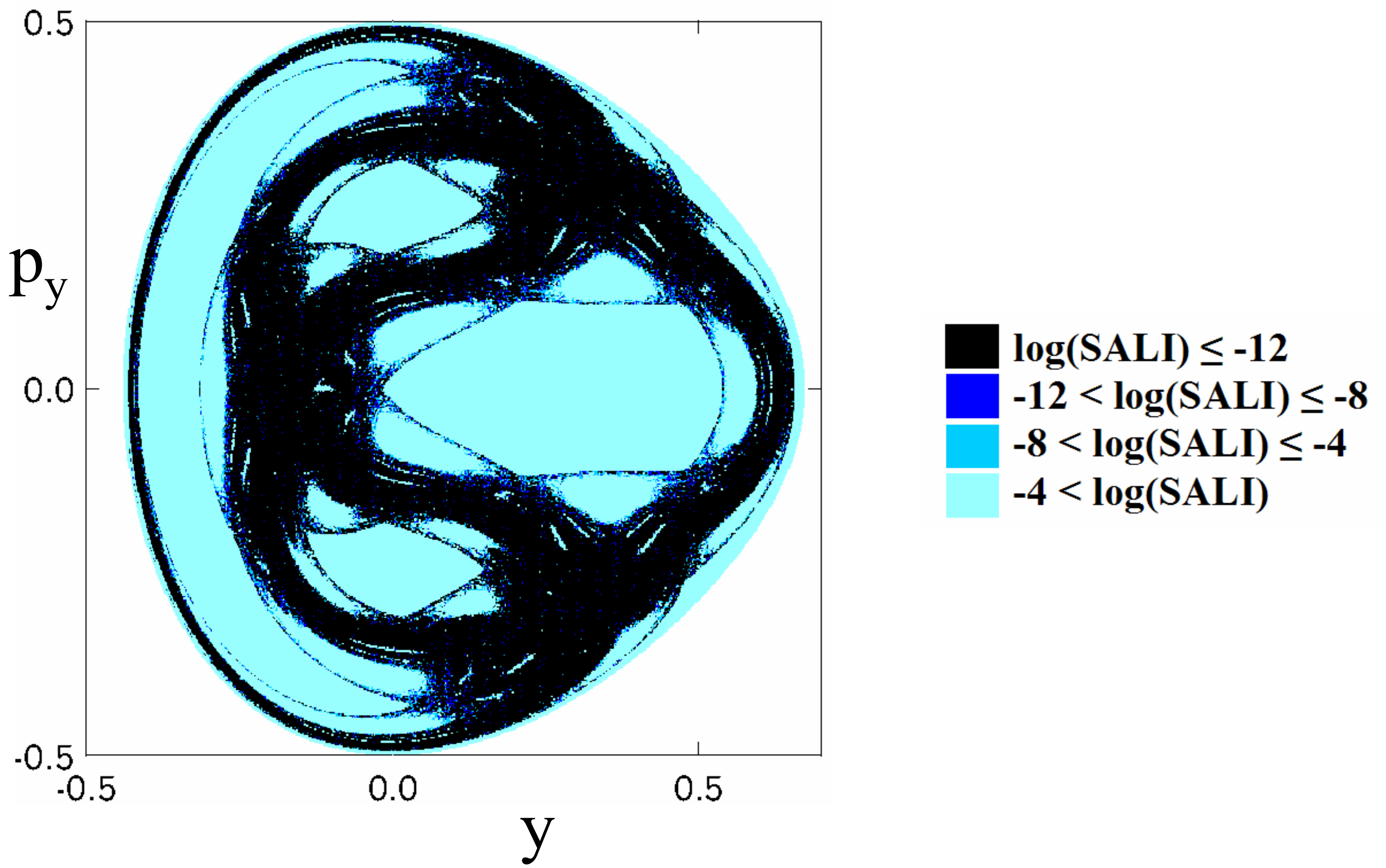


t=1000



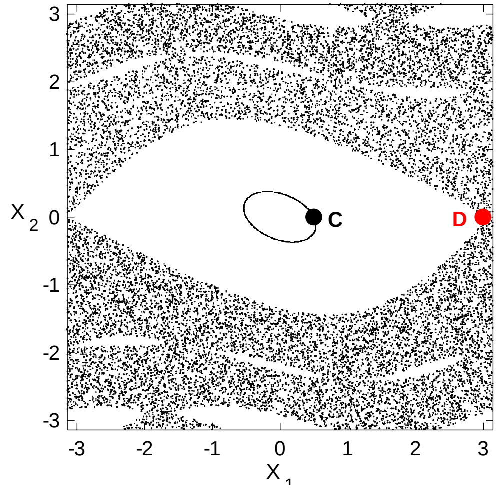
t=4000

Applications – Hénon-Heiles system



Applications – 4D map

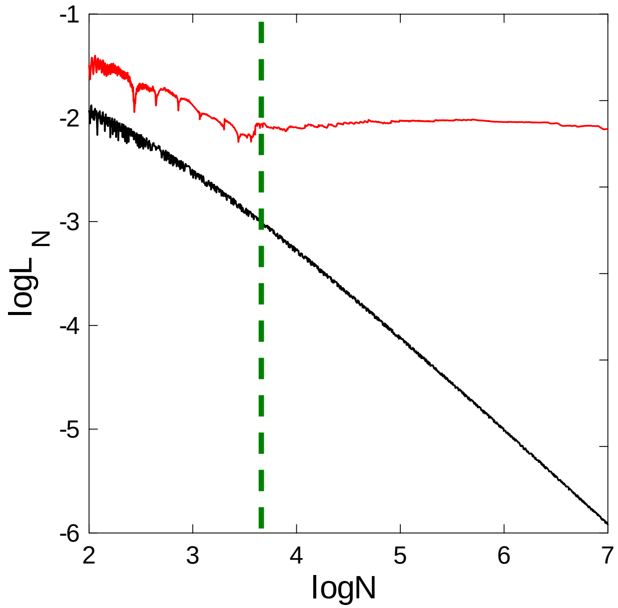
$$
\begin{aligned}
x_1' &= x_1 + x_2 \\
x_2' &= x_2 - \nu \sin(x_1 + x_2) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)] \\
x_3' &= x_3 + x_4 \\
x_4' &= x_4 - \kappa \sin(x_3 + x_4) - \mu\,[1 - \cos(x_1 + x_2 + x_3 + x_4)]
\end{aligned}
\qquad (\mathrm{mod}\ 2\pi)
$$

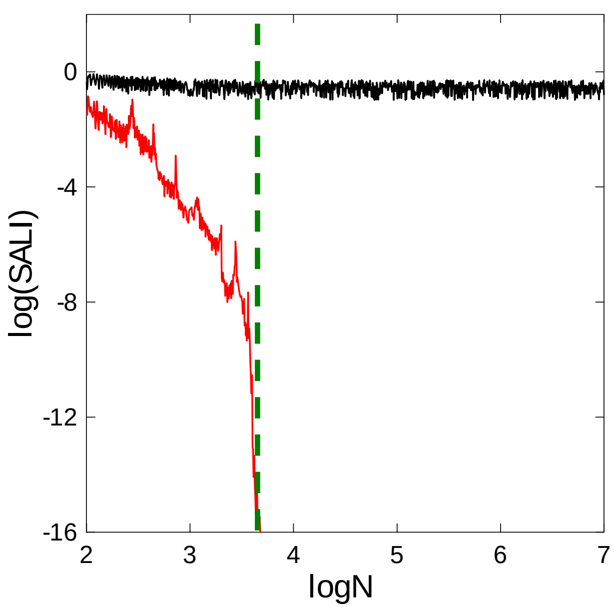


For ν=0.5, κ=0.1, μ=0.1 we consider the orbits:

ordered orbit C *with initial conditions* $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D *with initial conditions* $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.





Behavior of the SALI

Hamiltonian flows and multidimensional maps

The ordered motion occurs on a torus and two different initial deviation vectors become tangent to different directions on the torus.

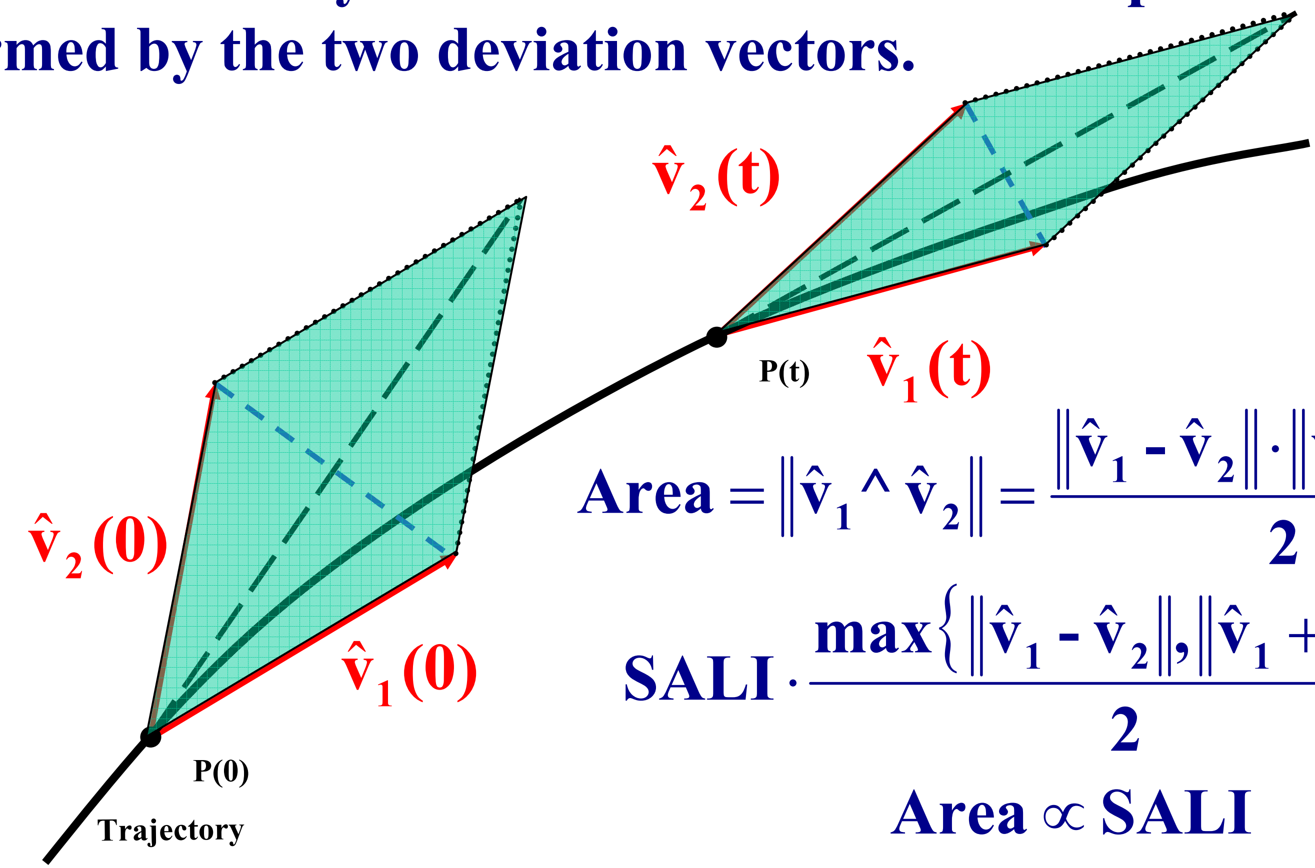
In chaotic cases two initially different deviation vectors tend to coincide to the direction defined by the most unstable nearby manifold.

SALI→0 for chaotic orbits (exponential decay)

SALI→constant ≠ 0 for regular orbits

Definition of Generalized Alignment Index (GALI)

SALI effectively measures the 'area' of the parallelogram formed by the two deviation vectors.



$$\text{Area} = \left\| \hat{v}_1 \wedge \hat{v}_2 \right\| = \frac{\left\| \hat{v}_1 - \hat{v}_2 \right\| \cdot \left\| \hat{v}_1 + \hat{v}_2 \right\|}{2} =$$

$$\text{SALI} \cdot \frac{\max\left\{ \left\| \hat{v}_1 - \hat{v}_2 \right\|, \left\| \hat{v}_1 + \hat{v}_2 \right\| \right\}}{2} \Rightarrow$$

$$\text{Area} \propto \text{SALI}$$

Definition of GALI

In the case of an N degree of freedom Hamiltonian system or a 2N symplectic map we follow the evolution of

k deviation vectors with $2 \leq k \leq 2N$,

and define (Skokos et al., 2007, Physica D, 231, 30) the Generalized Alignment Index (GALI) of order k :

$$GALI_k(t) = \left\| \hat{v}_1(t) \wedge \hat{v}_2(t) \wedge \ldots \wedge \hat{v}_k(t) \right\|$$

where

$$\hat{v}_1(t) = \frac{v_1(t)}{\left\| v_1(t) \right\|}$$

Computation of GALI

For k deviation vectors:

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_k \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1\,2N} \\ v_{21} & v_{22} & \cdots & v_{2\,2N} \\ \vdots & \vdots & & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{k\,2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix} = A \cdot \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \\ \hat{e}_{2N} \end{bmatrix}$$

the 'norm' of the wedge product is given by:

$$\left\| \hat{v}_1 \wedge \hat{v}_2 \wedge \cdots \wedge \hat{v}_k \right\| = \left\{ \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2N} \begin{vmatrix} v_{1\,i_1} & v_{1\,i_2} & \cdots & v_{1\,i_k} \\ v_{2\,i_1} & v_{2\,i_2} & \cdots & v_{2\,i_k} \\ \vdots & \vdots & & \vdots \\ v_{k\,i_1} & v_{k\,i_2} & \cdots & v_{k\,i_k} \end{vmatrix}^2 \right\}^{1/2} = \sqrt{\det(A \cdot A^T)}$$

Computation of GALI

From Singular Value Decomposition (SVD) of A^T we get:

$$A^T = U \cdot W \cdot V^T$$

where U is a column-orthogonal 2N×k matrix ($U^T \cdot U = I$), V^T is a k×k orthogonal matrix ($V \cdot V^T = I$), and W is a diagonal k×k matrix with positive or zero elements, the so-called singular values. So, we get:

$$\det(A \cdot A^T) = \det(V \cdot W^T \cdot U^T \cdot U \cdot W \cdot V^T) = \det(V \cdot W \cdot I \cdot W \cdot V^T) =$$

$$\det(V \cdot W^2 \cdot V^T) = \det(V \cdot \mathrm{diag}(w_1^2, w_2^2, \ldots w_k^2) \cdot V^T) = \prod_{i=1}^{k} w_i^2$$

Thus, $GALI_k$ is computed by:

$$GALI_k = \sqrt{\det(A \cdot A^T)} = \prod_{i=1}^{k} w_i \Rightarrow \log\left(GALI_k\right) = \sum_{i=1}^{k} \log\left(w_i\right)$$

Behavior of $GALI_k$ for chaotic motion

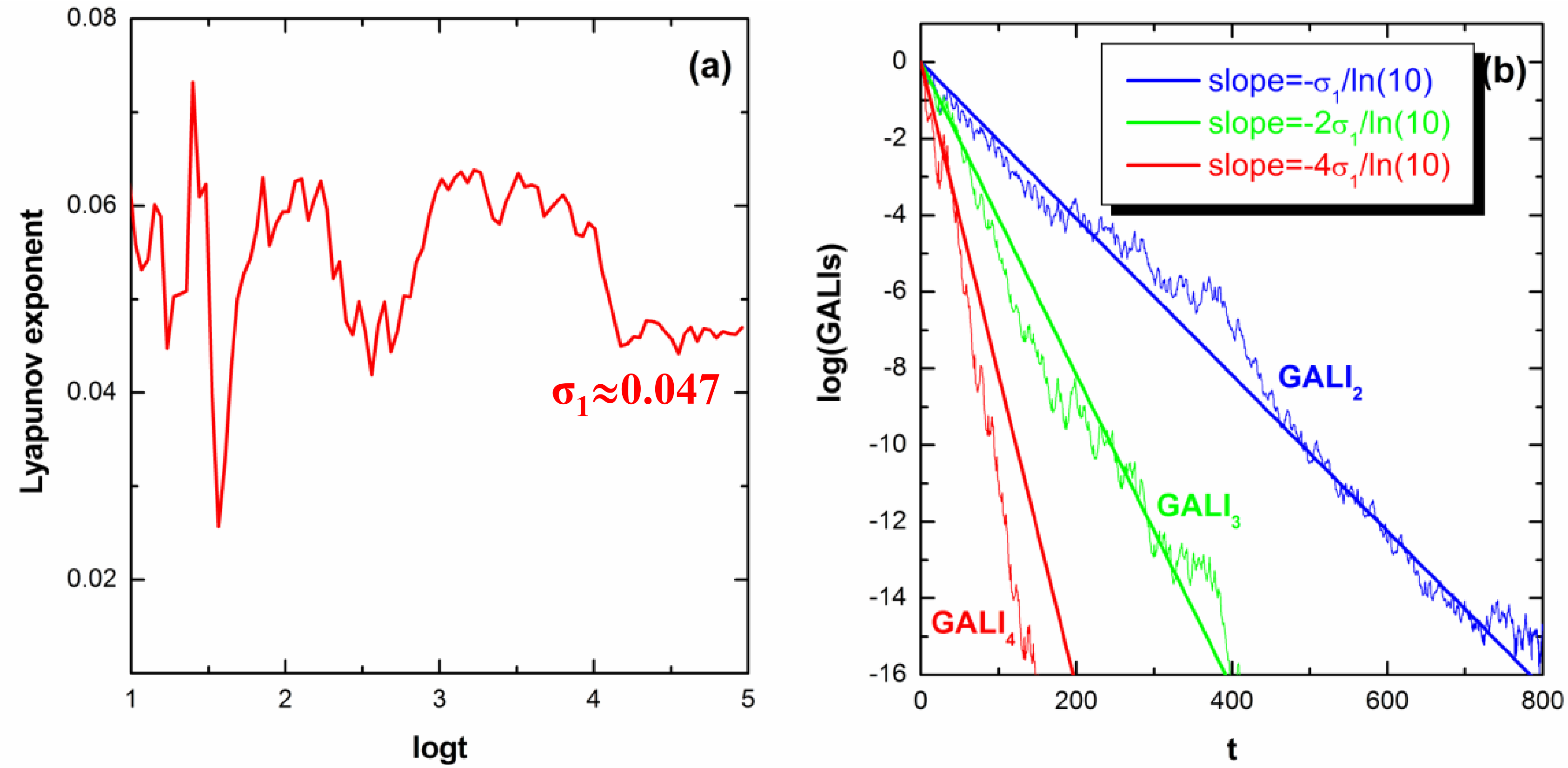
$GALI_k$ ($2 \leq k \leq 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \ldots, \sigma_k$:

$$GALI_k(t) \propto e^{-\left[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)\right]t}$$

The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of $GALI_k$ for chaotic motion

2D Hamiltonian (Hénon-Heiles system)

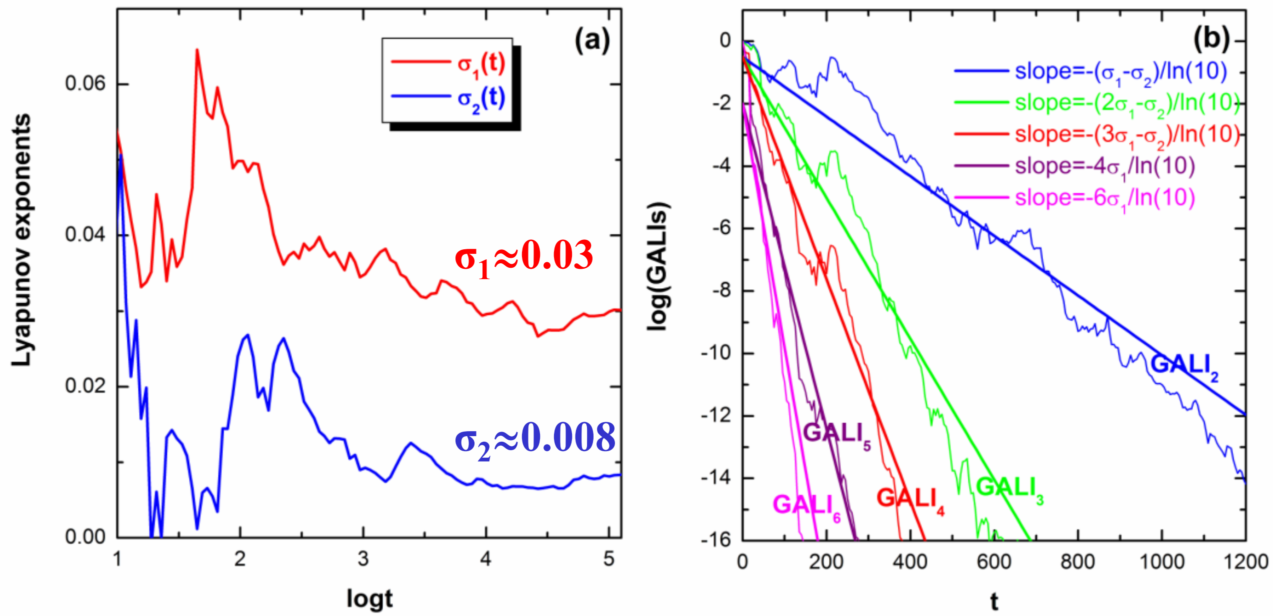


Behavior of $GALI_k$ for chaotic motion

3D system:

$$H_3 = \sum_{i=1}^{3} \frac{\omega_i}{2}(q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with $\omega_1=1$, $\omega_2=\sqrt{2}$, $\omega_3=\sqrt{3}$, $H_3=0.09$.

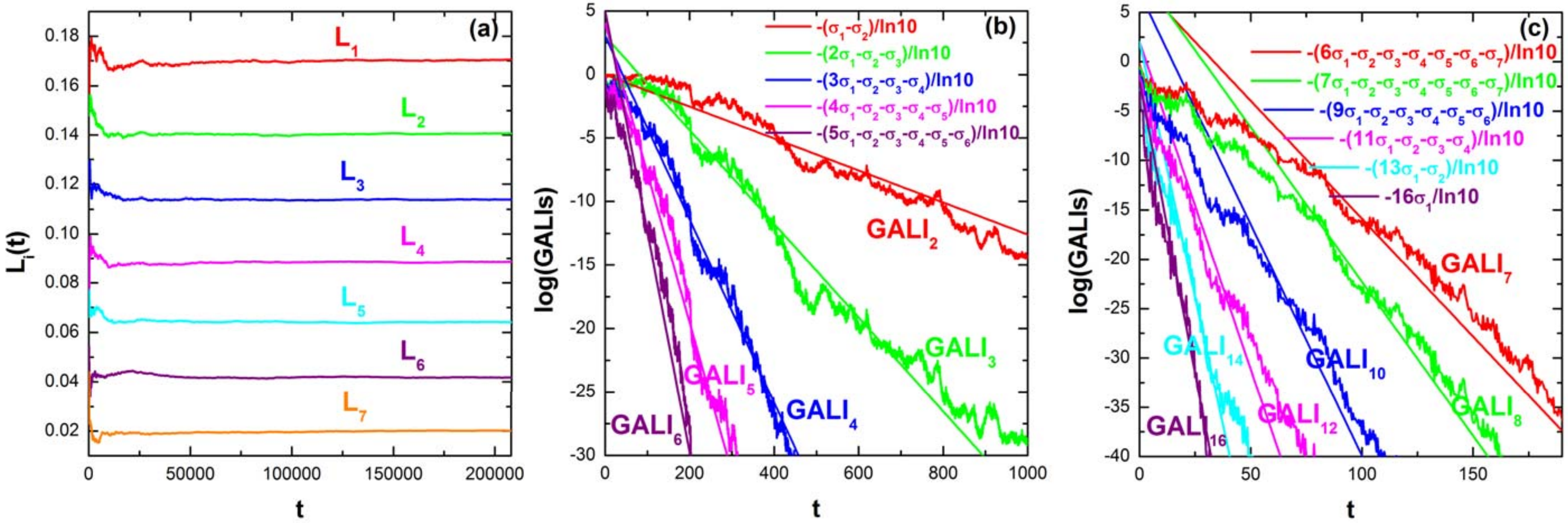


Behavior of $GALI_k$ for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=0}^{N}\left[\frac{1}{2}(q_{i+1}-q_i)^2 + \frac{\beta}{4}(q_{i+1}-q_i)^4\right]$$

with fixed boundary conditions, N=8 and β=1.5.



Behavior of $GALI_k$ for regular motion

If the motion occurs on an s-dimensional torus with s≤N then the behavior of $GALI_k$ is given by (Skokos et al., 2008, EPJ-ST, 165, 5):

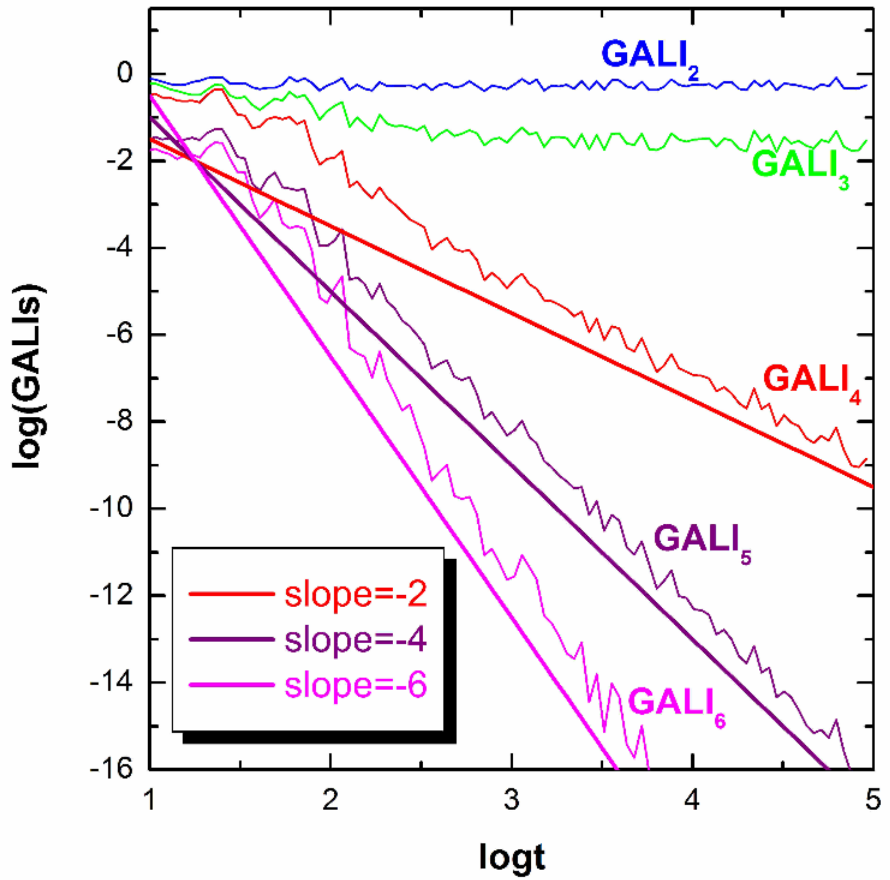
$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \le k \le s \\ \dfrac{1}{t^{k-s}} & \text{if} \quad s < k \le 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad 2N-s < k \le 2N \end{cases}$$

while in the common case with s=N we have :

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if} \quad 2 \le k \le N \\ \dfrac{1}{t^{2(k-N)}} & \text{if} \quad N < k \le 2N \end{cases}$$

Behavior of $GALI_k$ for regular motion

3D Hamiltonian



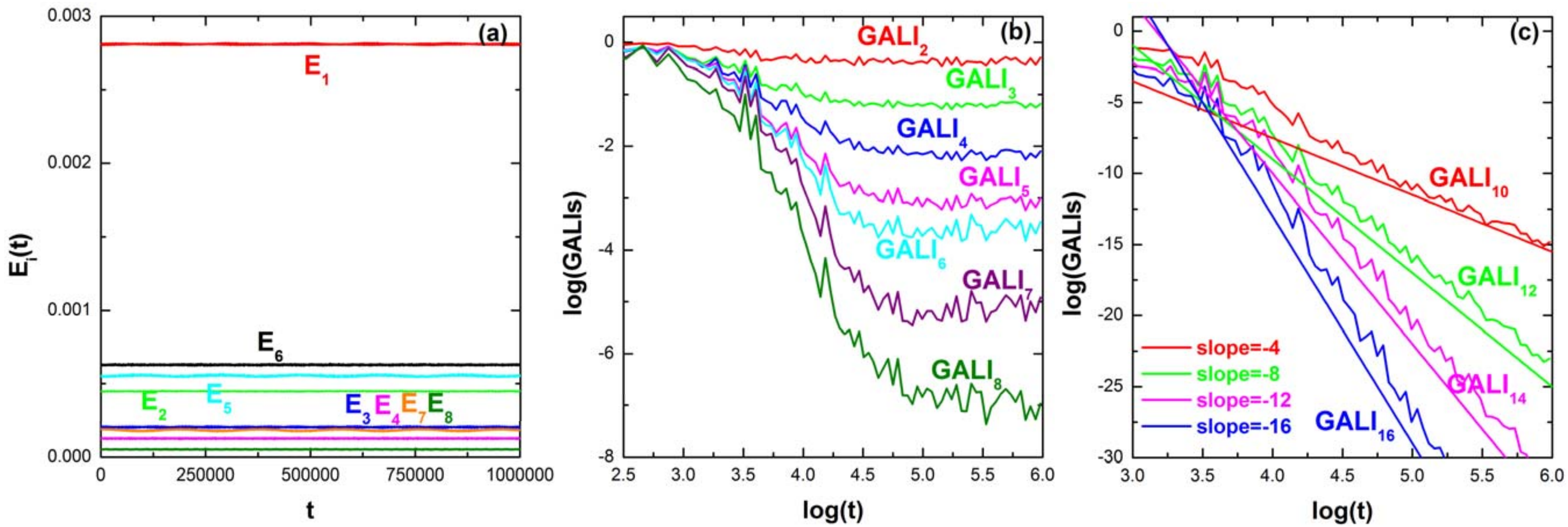
Behavior of $GALI_k$ for regular motion

N=8 FPU system: The unperturbed Hamiltonian (β=0) is written as a sum of the so-called **harmonic energies E_i**:

$$E_i = \frac{1}{2}\left(P_i^2 + \omega_i^2 Q_i^2\right),\ i = 1,\ldots,N$$

with:

$$Q_i = \sqrt{\frac{2}{N+1}}\sum_{i=1}^{N} q_i \sin\left(\frac{ki\pi}{N+1}\right),\ P_i = \sqrt{\frac{2}{N+1}}\sum_{i=1}^{N} p_i \sin\left(\frac{ki\pi}{N+1}\right),\ \omega_i = 2\sin\left(\frac{i\pi}{2(N+1)}\right)$$

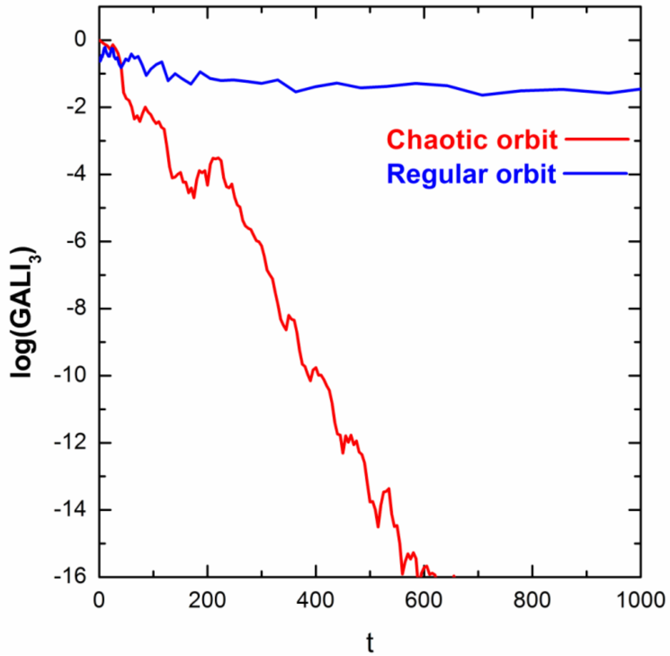


Global dynamics

- **GALI$_2$** (practically equivalent to the use of SALI)
- **GALI$_N$**

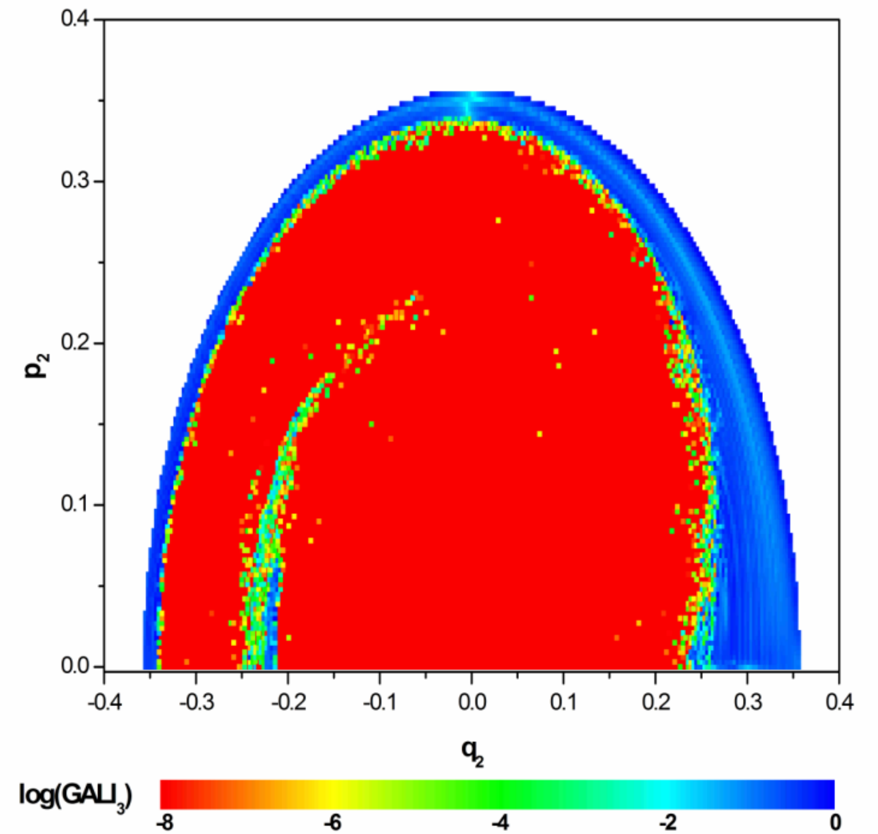
Chaotic motion: GALI$_N \rightarrow 0$ (exponential decay)

Regular motion: GALI$_N \rightarrow$ constant $\neq 0$



3D Hamiltonian

Subspace $q_3=p_3=0$, $p_2 \geq 0$ for $t=1000$.



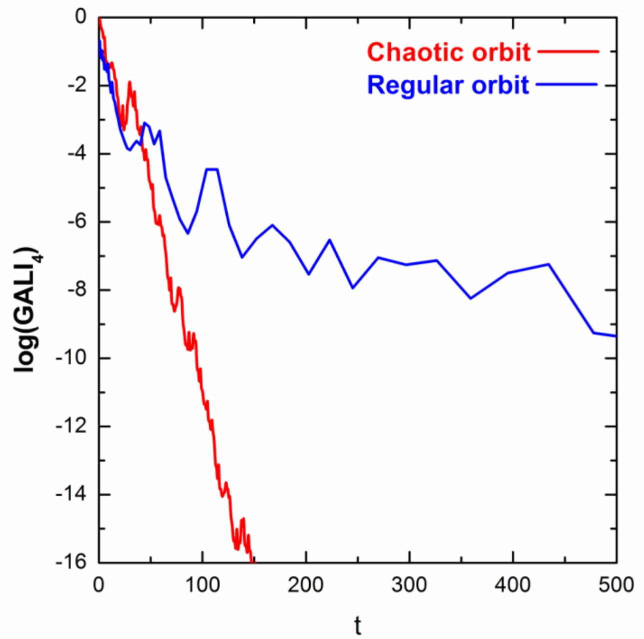
Global dynamics

GALI$_k$ with k>N

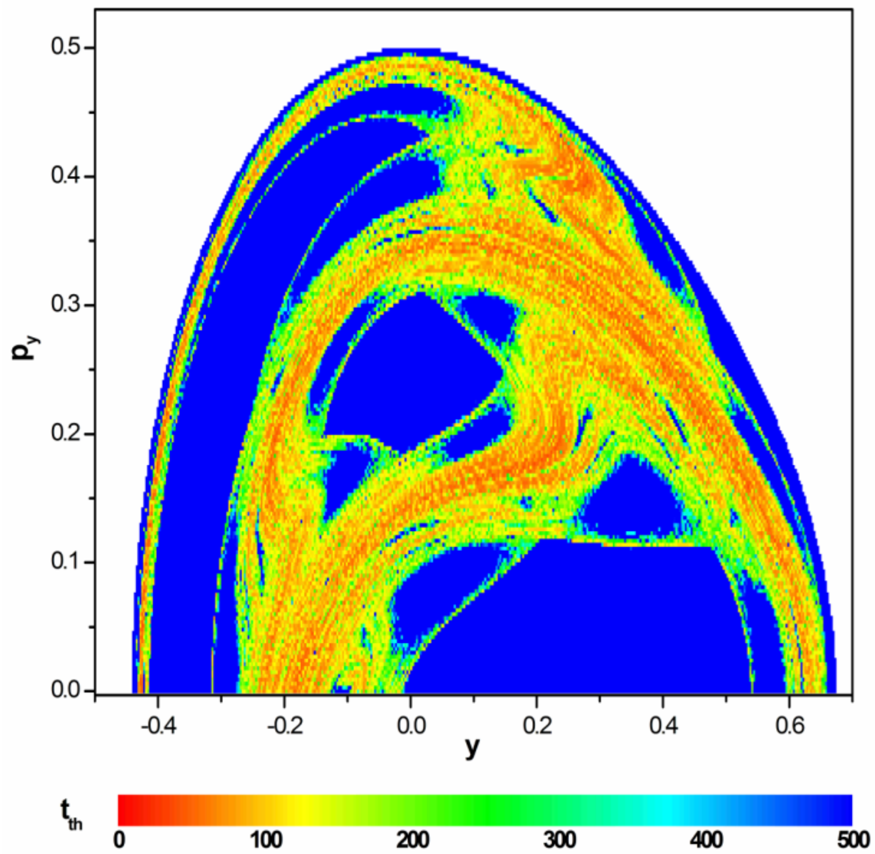
The index tends to zero both for regular and chaotic orbits but with completely different time rates:

Chaotic motion: exponential decay

Regular motion: power law

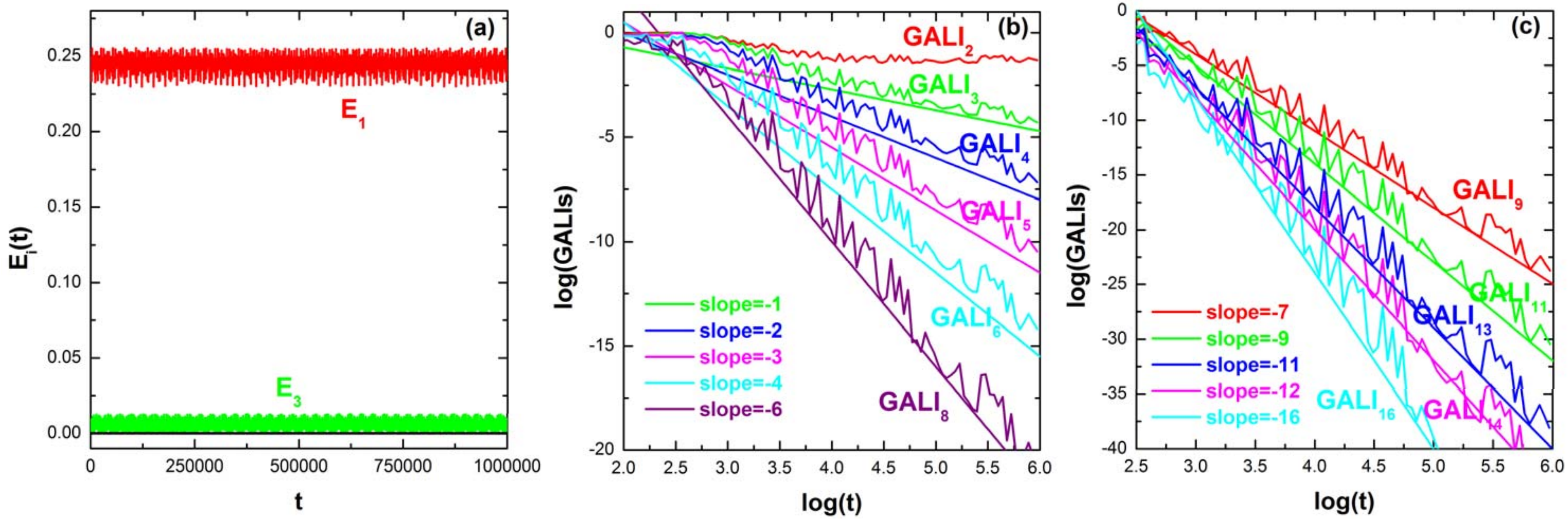


2D Hamiltonian (Hénon-Heiles)
Time needed for GALI$_4$<10^{-12}



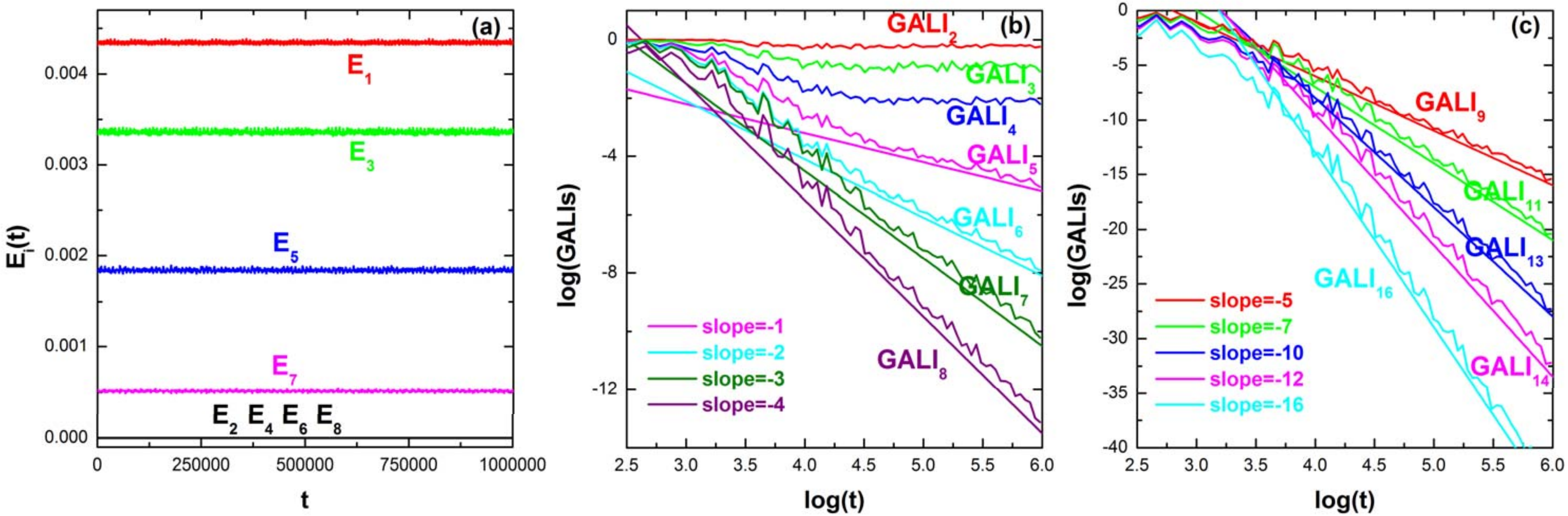
Regular motion on low-dimensional tori

A regular orbit lying on a 2-dimensional torus for the N=8 FPU system.



Regular motion on low-dimensional tori

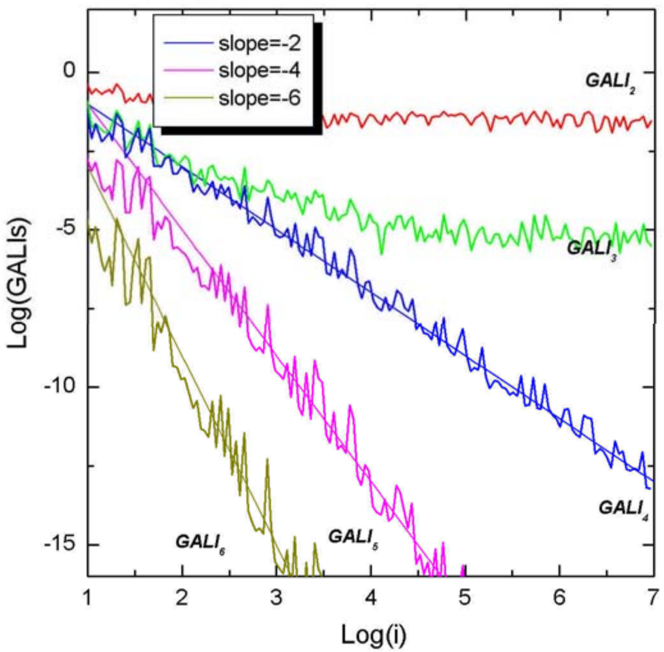
A regular orbit lying on a 4-dimensional torus for the N=8 FPU system.



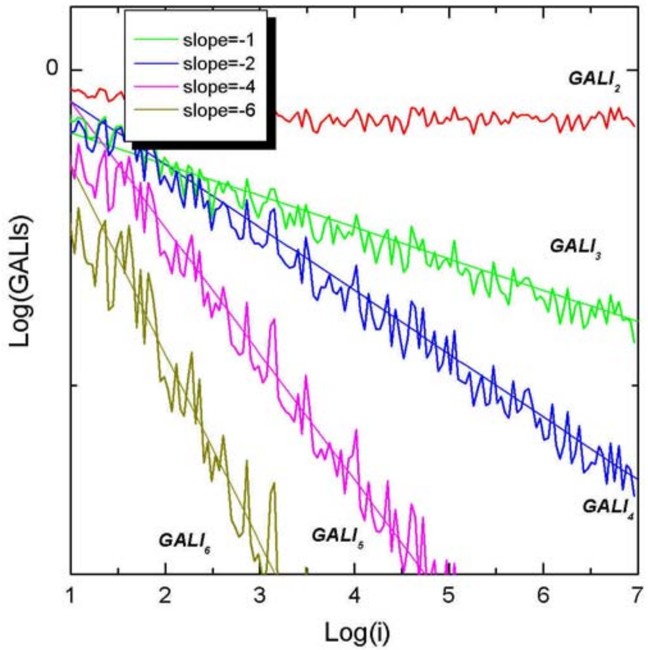
Low-dimensional tori - 6D map

$$
\left.\begin{aligned}
x'_1 &= x_1 + x'_2\\
x'_2 &= x_2 + \tfrac{K_1}{2\pi}\sin(2\pi x_1) - \tfrac{B}{2\pi}\left\{\sin[2\pi(x_5 - x_1)] + \sin[2\pi(x_3 - x_1)]\right\}\\
x'_3 &= x_3 + x'_4\\
x'_4 &= x_4 + \tfrac{K_2}{2\pi}\sin(2\pi x_3) - \tfrac{B}{2\pi}\left\{\sin[2\pi(x_1 - x_3)] + \sin[2\pi(x_5 - x_3)]\right\}\\
x'_5 &= x_5 + x'_6\\
x'_6 &= x_6 + \tfrac{K_3}{2\pi}\sin(2\pi x_5) - \tfrac{B}{2\pi}\left\{\sin[2\pi(x_3 - x_5)] + \sin[2\pi(x_1 - x_5)]\right\}
\end{aligned}\right\}\ (\text{mod } 1)
$$

3D torus



2D torus



Behavior of $GALI_k$

Chaotic motion:

$GALI_k \rightarrow 0$ exponential decay

$$GALI_k(t) \propto e^{-[(\sigma_1-\sigma_2)+(\sigma_1-\sigma_3)+\ldots+(\sigma_1-\sigma_k)]t}$$

Regular motion:

$GALI_k \rightarrow \text{constant} \neq 0$ or $GALI_k \rightarrow 0$ power law decay

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq s \\ \dfrac{1}{t^{k-s}} & \text{if } s < k \leq 2N-s \\ \dfrac{1}{t^{2(k-N)}} & \text{if } 2N-s < k \leq 2N \end{cases}$$

Conclusions

- **Generalizing the SALI method we define the Generalized ALignment Index of order k ($GALI_k$) as the volume of the generalized parallelepiped, whose edges are k unit deviation vectors. $GALI_k$ is computed as the product of the singular values of a matrix (SVD algorithm).**
- **Behaviour of $GALI_k$:**
  - ✓ **Chaotic motion: it tends exponentially to zero with exponents that involve the values of several Lyapunov exponents.**
  - ✓ **Reguler motion: it fluctuates around non-zero values for $2 \leq k \leq s$ and goes to zero for $s < k \leq 2N$ following power-laws, with s being the dimensionality of the torus.**
- **$GALI_k$ indices :**
  - ✓ **can distinguish rapidly and with certainty between regular and chaotic motion**
  - ✓ **can be used to characterize individual orbits as well as "chart" chaotic and regular domains in phase space.**
  - ✓ **are perfectly suited for studying the global dynamics of multidimentonal systems**
  - ✓ **can identify regular motion in low–dimensional tori**

References

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